Relative Value-at-Risk (Relative VaR)

[Nematrian website page: RelativeVaR, © Nematrian 2015]

Suppose we want to derive the (relative) portfolio Value-at-Risk (relative VaR) when returns $r_1, r_2, \dots, r_N$ on the N exposures are jointly Gaussian, assuming that the corresponding portfolio weights are $a_1, a_2, \dots, a_N$ and corresponding benchmark weights are $b_1, b_2, \dots, b_N$.

By jointly Gaussian we mean that the vector of returns $\mathbf{r} = (r_1, r_2, \dots, r_N)^T$ is distributed as a multivariate normal distribution $N(\boldsymbol{\mu}, \mathbf{V})$, where $\boldsymbol{\mu}$ is a vector of mean returns and $\mathbf{V}$ is a covariance matrix.

A property of any N-dimensional Gaussian, i.e. multivariate Normal, distribution that can be derived relatively simply from the probability density function of such a distribution is that if $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{V})$ and if we have a constant vector $\mathbf{c} = (c_1, \dots, c_N)^T$ then $\mathbf{c}.\mathbf{x} \equiv c_1 x_1 + \dots + c_n x_n$ is univariate Normal $N(\mu, \sigma^2)$ for some μ and σ^2. Specifically:

$$\mu = \mathbf{c}.\boldsymbol{\mu} = c_1 \mu_1 + \dots + c_n \mu_n$$
$$\sigma^2 = \mathbf{c}^T \mathbf{V} \mathbf{c} = \sum_{i=1}^{N} \sum_{j=1}^{N} c_i V_{ij} c_j$$

where the V_{ij} are the elements of the covariance matrix $\mathbf{V}$.

By relative return we mean the return on the portfolio relative to the return on the benchmark. For any given time period the return on the portfolio is the weight the portfolio ascribes to the exposure times the return on that exposure, i.e. is $a_1 r_1 + \dots + a_N r_N = \mathbf{a}.\mathbf{r}$, where $\mathbf{a}$ is a vector with components a_i. Likewise the return on the benchmark is is $b_1 r_1 + \dots + b_N r_N = \mathbf{b}.\mathbf{r}$, where $\mathbf{b}$is a vector with components b_i. So the relative return* is $(\mathbf{a} - \mathbf{b}).\mathbf{r}$. Hence the relative return is distributed as a univariate Normal distribution with $\mu = (\mathbf{a} - \mathbf{b}).\boldsymbol{\mu}$ and $\sigma^2 = (\mathbf{a} - \mathbf{b})^T \mathbf{V} (\mathbf{a} - \mathbf{b})$. The relative VaR with confidence level α is then $VaR_\alpha = \mu + \sigma N^{-1}(1 - \alpha)$ with these μ and σ, where $N^{-1}(x)$ is the inverse Normal function.

* N.B. there is an assumption here that the returns over the relevant time interval are small, otherwise there is an issue about whether to use arithmetic relative or geometric relatives etc., see e.g. Relative Return Computations).