Performance Measurement Theory

Abstract

The aim of the attached pages is to summarise some of the theory behind investment performance measurement and attribution.

Contents

1. Introduction

[PerformanceMeasurementTheory1]

1.1 The main purpose of investment performance measurement and attribution is to determine in a quantitative sense how well a portfolio has performed and where that performance has come from. Mathematically, performance measurement is relatively straightforward compared with risk measurement, although careful attention to accounting detail is required. Different audiences may want to see attribution subdivisions in different ways. Because results are often highly sensitive to the accuracy of input data, it can also provide a useful check of the accuracy of the underlying accounting process. Performance attribution involves calculating the total returns for both fund and benchmark (for the relevant period), creating suitably accurate models of how these total returns can be built up from the various constituent parts, and then decomposing the differences in ways that are illuminating to the various audiences. For a hedge fund or a trading account, there might be no explicit benchmark as such, so performance attribution might instead concentrate on a cash benchmark.

1.2 The modelling process will subdivide time into various periods. Returns do not compound additively over time, but geometrically. The root time period can be as short as a single day, although such a short period can create extra work without necessarily offering any material improvement in accuracy. Even over very short periods it may be necessary to make assumptions or approximations, or equivalently you may have to accept that there will be residuals that need explaining or quantifying.

1.3 Ideally any performance attribution should start with the contributions to performance arising from each individual line of stock for both the fund and the benchmark. These would then be grouped together in some suitable fashion, e.g. a country/sector classification/portfolio design structure (for equity and managed funds) and/or using *factor* exposures such as duration (for bond funds). This may involve a hierarchical structure, drilling down potentially several levels. Sometimes cash is kept separate, and sometimes aggregated with the rest of the portfolio. Security classifications need to be maintained (including relevant factor exposures). The classification of a given security and its factor exposures may change over time. If the portfolio contains derivatives or similar instruments their values may need to be divided between two or more characteristics/factors simultaneously, often positive to one characteristic/factor and negative to another, see e.g. Kemp (1997), LIFFE (1992a) or LIFFE (1992b). Carrying out the same calculations for large numbers of funds simultaneously is facilitated by giving careful consideration to how to store all of this data in a suitable fashion, and how to process it efficiently. Many of the same data management issues also arise in practical risk management systems.

2. Mathematics of multi-period analysis

[PerformanceMeasurementTheory2]

2.1 Suppose that we are interested in calculating the rate of return on a portfolio from time 0 to time 1 using some suitable units of time. Suppose that there are *n* new money payments into or out of the portfolio in the period of value C_j (positive for inflows, negative for outflows) occurring at times t_j for $j = 1, \dots, n$. The t_j are assumed to be ordered so that $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$. The market values at the corresponding points in time (immediately after receipt of the new money) are M_j. Dividend/interest payments are treated as outflows from the relevant stock/bond sector and inflows into the cash sector, and so net to zero at the total fund level (unless the income is paid away).

2.2 The *time-weighted rate of return* for the period is then $g = \prod_1^n (1 + g_j) - 1$ where $1 + g_j = (M_j - C)/M_{j-1}$. The time-weighted rate of return is effectively equivalent to the growth in a unit net asset value price (were the fund to be unitised and were it to accumulate income internally, ignoring complications such as bid/offer spreads, etc.) The positive or negative impact of money arriving or being withdrawn from the portfolio at opportune or inopportune times is stripped out of the calculation. Time-weighted rates of return naturally compound up over time, i.e. if the time-weighted rate of return in one period is g_a and in the next is g_b then the time weighted rate of return for the combined period is g where $1 + g = (1 + g_a) \times (1 + g_b)$. This also means that 'logged' returns, i.e. $\log(1 + g)$, naturally add up through time, i.e. $\log(1 + g) = \log(1 + g_a) + \log(1 + g_b)$.

2.3 The *money-weighted* or *internal rate of return* on a fund over the same period, is defined as the 'sensible' solution for r to the following equation (if the C_j are of differing signs then there will usually be more than one solution, although normally only one would be intrinsically 'sensible'):

$$M_{start}(1 + r) + \sum_{j=1}^{n} C_j (1 + r)^{1-t_j} = M_{end}$$

2.4 The time-weighted rate of return and the money-weighted rate of return are thus the same if there have been no cash flows during the period.

2.5 One nearly always assumes that $(1 + r)^t \approx 1 + tr$. The internal rate of return can therefore be approximated by the formula $r = CR/MF$, where the *contribution to return* (CR), *net new money* (NNM), *time-weighted net investment* ($TWNI$), and *mean fund* (MF), are defined as follows:

$$CR = M_{end} - M_{start} - NNM$$

$$NNM = \sum_{j=1}^{n} C_j$$

$$MF = M_{start} + TWNI$$

$$TWNI = \sum_{j=1}^{n} C_j (1 - t_j)$$

2.6 The internal rate of return is the (constant) interest rate that a bank account would need to provide (possibly negative) to return the same amount at the end of the period as the portfolio, given the same new money flows and the same start market value. It is therefore the same as the money-weighted rate of return and so again does not naturally compound up over time.

2.7 Calculating the time-weighted rate of return in principle involves valuations whenever there is a cash flow. This can be time consuming, unless you have an exceptionally good valuation engine (and even then is potentially impossible if you wish to value at the exact intra-day point of time at which a particular trade takes place).

2.8 In practice, therefore, performance measurers often merely chain-link internal rates of return. This is because the money weighted and time weighted rates of return are the same if there are no intra-period new money flows. So, if you calculate internal rates of return sufficiently often and chain-link them together then the result will always tend to the time-weighted rate of return.

In certain other special circumstances, the money-weighted and time-weighted rates of return are also identical. Normally, cash flows and market values will be expressed in some base currency, but suppose we generalise the calculation of money-weighted rates of return so that it can include an arbitrary *calculation numeraire*, which is worth f_j in the base currency at time t_j The money-weighted rate of return then becomes r where $(1+r)=(1+s)\times f_n/f_0$ and where s is the solution to:

$$\frac{M_{start}}{f_0}(1+s)+\sum_{j=1}^{n}\frac{C_j}{f_j}(1+s)^{1-t_j}=\frac{M_{end}}{f_n}$$

2.9 The money-weighted rate of return described above is then merely a special case of this calculation with a constant (in base currency) numeraire. Suppose that we choose $f_j=\left(1+g_j\right)$, where g_j corresponds to the true cumulative time-weighted return from time 0 to time t_j Then $s=0$ and the money-weighted rate of return, r, will (in this numeraire) be *identical* to the time-weighted rate of return g. If f_j closely approximates to $\left(1+g_j\right)$ then s will closely approximate to 0, and the approximation $(1+s)^t\approx 1+ts$ will be very good. The money-weighted rate of return using such a numeraire will then be very similar to the true time-weighted rate of return. If the new money flows are small in relation to start *and* end market values then the money-weighted rate of return will also be very similar to the true time-weighted rate of return, irrespective of the calculation numeraire.

2.10 The calculation numeraire can be differentiated from the *presentation numeraire* used to express the results of the calculation, which will normally be the base currency of the portfolio. If the presentation numeraire is h_j then the rates of return would be restated to be a_j where $(1+a)=(1+r)\times h_0/h_n$.

2.11 The above approach requires not only fund holding and valuation price data but also information on the prices at which individual transactions were carried out. If these are difficult to obtain then an alternative, less exact, methodology involves *buy and hold* attribution. In this methodology, the return on each line of stock is imputed merely from market data over a given period (usually daily) on the assumption that no transactions have taken place. Such an approach produces the same answer as a true transactions-based analysis either if no transactions occur or if they occur at the prices assumed in the algorithm. Unfortunately this approximation can lead to significant residuals for funds with high turnover or subject to significant dealing costs.

3. Performance attribution

[PerformanceMeasurementTheory3]

3.1 Portfolios will typically contain several sectors, in which case, given the same linear approximation as used above, the total fund and benchmark returns, r and R will be as follows, w_i = mean fund weighting for sector i, r_i = return for that individual sector etc, b_i = benchmark weighting for sector i and q_i = return on benchmark for sector i:

$$r = \frac{CR}{MF} = \sum_i w_i r_i$$

$$R = \sum_i b_i q_i$$

where $r_i = CR_i/MF_i$ and $w_i = MF_i/MF$

3.2 Their difference is therefore as follows (since $\sum_i w_i = \sum_i b_i = 1$):

$$r - R = \sum_i (w_i - b_i)(q_i - R) + \sum_i b_i(r_i - q_i) + \sum_i (w_i - b_i)(r_i - q_i)$$

3.3 We can rewrite the relative performance as follows:

$$r - R = \sum_i AA_i + \sum_i SS_i + \sum_i IE_i$$

where $AA_i = \sum_i (w_i - b_i)(q_i - R)$, $SS_i = \sum_i b_i(r_i - q_i)$ and $IE_i = \sum_i (w_i - b_i)(r_i - q_i)$.

3.4 The AA_i are the contributions from 'asset allocation', the SS_i are the contributions from 'stock selection' and the IE_i are the contributions from an 'interaction effect'. The interaction effect is the cross-product term that arises from the fact that the value added by stock selection is based on the amount of assets involved. Typically, the interaction effect is added into stock selection if you are a 'top-down' manager and into asset allocation if you are a 'bottom-up' manager.

3.5 The above analysis concentrates on *additive* attribution. To make the contributions from asset allocation and stock selection chain-link they can be restated in a geometric fashion as follows: $GAA_i = (1+g)^{AA_i/ARR} - 1$ and $GSS_i = (1+g)^{SS_i/ARR}$ (and a corresponding GIE_i if the interaction term is kept separate) where g = geometric relative return at total assets level, ARR = additive relative return at total assets level and AA_i and SS_i are the additive asset allocation contribution and additive stock selection contribution from sector i. Or, one can use natural logarithms using, say, $LAA_i = AA_i \times \log(1+g)/ARR$ so that $GAA_i = \exp(LAA_i) - 1$. The total logarithmic contribution to return from a particular source over several periods can then be found merely by adding these terms together over.

3.6 Decomposing returns by 'factors' is conceptually quite similar. However, we also need:

(a) For both fund and benchmark, the average exposure to each factor involved in the decomposition, say $a_{fd,i,1}, a_{fd,i,2}, \ldots$ and $a_{bmk,i,1}, a_{bmk,i,2}, \ldots$

(b) For the benchmark only, the return a sector would deliver with zero factor exposure, $z_{i,0}$ and the extra return for a unit exposure to each individual factor (for each sector), say: $z_{i,1}, z_{i,2}, \ldots$ so that $R_i = z_{i,0} + \sum_{k=1} a_{bmk,i,k} z_{i,k}$.

3.7 The relative return can then be decomposed into:

$$r - R = \sum_i (w_i - b_i)(q_i - R) + \sum_{k \geq 1, i} w_i \left(a_{fd,i,k} - a_{bmk,i,k}\right) z_{i,k} + \sum_i w_i \left(r_i - \left(z_{i,0} + \sum_{k \geq 1} a_{fd,i,k} z_{ik} \right) \right)$$

3.8 The first term is the contribution from asset allocation, the second the component of the stock selection explained by the various factors, and the third the residual component of stock selection which is unexplained by the various factors. The second term would normally be shown decomposed by both sector and factor. The sector analysis described above is a special case with $z_{i,k} = 0$ and with more than one value for k. We would ideally want to build up the $a_{fd,i,k}$ by calculating the corresponding factor exposures by line of stock and then aggregating to the sector level. We might also do this for the benchmark as well or we might use a separate summarised data source.

3.9 Currency effects can be accommodated within this framework by including as separate 'sectors' any currency hedges away from the fund's base position. If the base position is a hedged benchmark then these would be reverse hedges to reintroduce exposure to that currency. Performance measurers have developed lots of other ways of taking currency into account, although many only seem particularly relevant for certain ways in which currency decisions might be taken vis-à-vis sector or security selection decisions.

Relevant Nematrian Web service tools

[PerformanceMeasurementTheoryTools]

Details of the main web service functions that the Nematrian website currently makes available that are related to performance measurement activities are set out here.

References

[PerformanceMeasurementTheoryRefs]

Kemp, M.H.D. (1997). Actuaries and derivatives. *British Actuarial Journal*, 3, pp. 51-162

LIFFE (1992a). The reporting and performance measurement of financial futures and options in investment portfolios. *The London International Financial Futures and Options Exchange*.

LIFFE (1992b). Futures and options: standards for measuring their impact on investment portfolios. *The London International Financial Futures and Options Exchange*.