Marginal Tail Value-at-Risk (Marginal TVaR) when underlying distribution is multivariate normal

[Nematrian website page: MarginalTVaRMultivariateNormal, © Nematrian 2015]

Suppose we have a set of n risk factors which we can characterise by an n-dimensional vector $\mathbf{x} = (x_1, \dots, x_n)^T$. Suppose that the (active) exposures we have to these factors are characterised by another n-dimensional vector, $\mathbf{a} = (a_1, \dots, a_n)^T$. The aggregate exposure is then $\mathbf{a}.\mathbf{x}$.

The *Value-at-Risk*, $VaR_\alpha(\mathbf{a})$, of the portfolio of exposures $\mathbf{a}$ at confidence level α, is defined as the

$$TVaR_\alpha = -\frac{1}{1-\alpha}\int_{-\infty}^{-VaR_\alpha} xf(x)dx$$

The Marginal Tail Value-at-Risk, $MTVaR_{\alpha,i}(\mathbf{a})$, is the sensitivity of $TVaR_\alpha(\mathbf{a})$ to a small change in i'th exposure. It is therefore:

$$MTVaR_{\alpha,i}(\mathbf{a}) = \frac{\partial TVaR_\alpha(\mathbf{a})}{\partial a_i}$$

In the case where the risk factors are multivariate normally distributed with mean $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ and covariance matrix V whose elements are V_{ij} we have $\mathbf{x} \sim N(\boldsymbol{\mu}, \mathbf{V})$ and hence $\mathbf{a}.\mathbf{x} \sim N(\mathbf{a}.\boldsymbol{\mu}, \mathbf{a}^T\mathbf{V}\mathbf{a})$. Hence $VaR_\alpha(\mathbf{a}) = -\left(\mathbf{a}.\boldsymbol{\mu} + \sigma N^{-1}(1-\alpha)\right)$.

Given the formula for the truncated first moments of a normal distribution we have:

$$TVaR_\alpha = -\frac{1}{1-\alpha}\int_{-\infty}^{-VaR_\alpha} xf(x)dx = -\frac{1}{1-\alpha}\left(\mu N\left(\frac{-VaR_\alpha - \mu}{\sigma}\right) - \sigma\phi\left(\frac{-VaR_\alpha - \mu}{\sigma}\right)\right)$$

where $\mu = \mathbf{a}.\boldsymbol{\mu}$, $\sigma \equiv \sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}}$, $N(x)$ is the (standard) normal cumulative distribution function and $\phi(x)$ is the (standard) normal probability density function.

Hence:

$$TVaR_\alpha = -\frac{1}{1-\alpha}\left(\mu N\left(\frac{\sigma N^{-1}(1-\alpha)}{\sigma}\right) - \sigma\phi\left(\frac{\sigma N^{-1}(1-\alpha)}{\sigma}\right)\right)$$

$$\Rightarrow TVaR_\alpha = -\frac{1}{1-\alpha}\left(\mu(1-\alpha) - \sigma\phi\left(N^{-1}(1-\alpha)\right)\right) = -\mu + \frac{\sigma}{1-\alpha}\sigma\phi\left(N^{-1}(1-\alpha)\right)$$

$$\Rightarrow MTVaR_{\alpha,i}(\mathbf{a}) \equiv \frac{\partial TVaR_\alpha(\mathbf{a})}{\partial a_i} = \frac{\partial}{\partial a_i}\left(-\mathbf{a}.\boldsymbol{\mu} + \frac{\phi\left(N^{-1}(1-\alpha)\right)}{1-\alpha}\sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}}\right)$$

$$\Rightarrow MTVaR_{\alpha,i}(\mathbf{a}) = \frac{\partial}{\partial a_i}\left(-\sum_{j=1}^{n} a_j\mu_j\right) + \frac{\phi\left(N^{-1}(1-\alpha)\right)}{1-\alpha}\frac{1}{2\sqrt{\mathbf{a}^T\mathbf{V}\mathbf{a}}}\frac{\partial}{\partial a_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} a_j V_{jk} a_k\right)$$

$$\Rightarrow MTVaR_{\alpha,i}(\mathbf{a}) = -\mu_i + \frac{\phi\left(N^{-1}(1-\alpha)\right)}{1-\alpha}\frac{1}{\sigma}\left(\sum_{j=1}^{n} a_j V_{ij}\right)$$

The second of these terms can be expressed in terms of the correlation between x_i and $\mathbf{a}.\mathbf{x}$ in a manner similar to Marginal VaR when underlying distribution is multivariate normal.

As risks arising from individual positions interact there is no universally agreed way of subdividing the overall risk into contributions from individual positions. However, a commonly used way is to define the *Contribution to Tail Value-at-Risk*, c_i, of the i'th position, a_i to be as follows:

$$c_i = a_i MTVaR_{\alpha,i}(\mathbf{a})$$

Conveniently the c_i then sum to the overall VaR:

$$\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i MTVaR_{\alpha,i}(\mathbf{a}) = \sum_{i=1}^{n} \left(-a_i \mu_i + \frac{\phi\left(N^{-1}(1-\alpha)\right)}{1-\alpha} \frac{1}{\sigma} \left(a_i \sum_{j=1}^{n} a_j V_{ij} \right) \right)$$

$$\Rightarrow \sum_{i=1}^{n} c_i = -\mathbf{a}.\boldsymbol{\mu} + \frac{\phi\left(N^{-1}(1-\alpha)\right)}{1-\alpha} \frac{\sigma^2}{\sigma} = -\mathbf{a}.\boldsymbol{\mu} + \sigma \frac{\phi\left(N^{-1}(1-\alpha)\right)}{1-\alpha} = TVaR_{\alpha}(\mathbf{a})$$

The property that the contributions to risk add to the total risk is a generic feature of any risk measure that is (first-order) homogeneous, a property that Tail Value-at-Risk exhibits.