Integration of Piecewise Polynomials against a Gaussian probability density function

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In a number of financial contexts it can be important to calculate the following integral:

$$\int_{-\infty}^{\infty} F(x) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

For example, this integral is relevant to calculating moments of a fat-tailed distribution, i.e. one whose quantile-quantile plot versus the Normal distribution, $F(x)$, is not unity. The faster $|F(x)|$ increases as $x \to \pm\infty$ the greater is the impact of fat-tailed behaviour, i.e. deviations from a density function of the form $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Thus risk measures such as expected shortfall (effectively a first moment computation, in which the leading element of $F(x)$ is of order $x^1 = x$) are more sensitive to fat-tailed effects than Value-at-Risk risk measures (effectively a zero moment computation, in which the leading element of $F(x)$ is of order x^0).

This integral can also appear in derivative pricing analysis, if payoff is being approximated by a piecewise polynomial function and the movement of the underlying is of a certain type (but see Valuing polynomial payoffs in a Black Scholes World, which suggests that some modification may be needed to cater for exponentials arising when converting the partial differential equation arising under a Gauss-Weiner process to one with 'standard' parabolic form).

Hence, it becomes helpful to be able to calculate the following integral rapidly:

$$V(z, \{a_0, \dots, a_n\}) = \int_0^z \sum_{i=0}^{n} a_i x^i \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

If z is large then we note that:

$$V(z, \{a_0, a_1, \dots, a_{2m-1}\}) = A + \frac{B}{\sqrt{2\pi}} - \frac{e^{-\frac{z^2}{2}}(C + D + E)}{\sqrt{2\pi}}$$

Where

$$A = \left(N(z) - \frac{1}{2}\right)\left(a_0 + \sum_{j=1}^{m-1}\left(\prod_{k=1}^{j}(2k-1)\right)a_{2j}\right)$$

$$B = a_1 + \sum_{j=1}^{m-1}\left(\prod_{k=1}^{j}(2k)\right)a_{2j+1}$$

$$C = \sum_{i=0}^{2m-2} z^i a_{i+1}$$

$$D = \sum_{j=0}^{m-1} z^{2j} \left(\sum_{k=1}^{m-1-j} \left(\prod_{p=1}^{k} (2j + 2p) \right) a_{2j+2k+1} \right)$$
$$E = \sum_{j=0}^{m-2} z^{2j+1} \left(\sum_{k=1}^{m-2-j} \left(\prod_{p=1}^{k} (2j + 2p + 1) \right) a_{2j+2k+2} \right)$$

If z is small then for higher order coefficients the above computation runs into machine rounding problems. It is then better to use a Taylor series expansion, bearing in mind that:

$$N(z) - \frac{1}{2} = \frac{1}{2} \operatorname{erf}\left(\frac{\sqrt{2}z}{2} \right)$$
$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(z) = z - \frac{z^3}{1! \times 3} + \frac{z^5}{2! \times 5} - \frac{z^7}{3! \times 7} + \frac{z^9}{4! \times 9} - \frac{z^{11}}{5! \times 11} + \cdots$$

Hence (if n is integral and $n \geq 0$) we have:

$$\int_0^z x^n e^{-\frac{x^2}{2}} dx = \sum_{j=0}^{\infty} \frac{(-1)^j z^{n+1+2j}}{2^j j! \, (n + 1 + 2j)}$$