Showing that a Gaussian copula is not in general an Archimdean copula

[Nematrian website page: ERMMTGaussianCopulaNotArchimedean, © Nematrian 2015]

An n-dimensional Archimedean copula is one that can be represented by:

$$C(u_1, u_2, \dots, u_n) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots + \psi^{-1}(u_n)\right)$$

One way of showing that the Gaussian copula is not in general an Archimedean copula is to consider a three dimensional Gaussian copula. Its copula density (for a correlation matrix Q) can be written as:

$$c_Q(u) = \frac{1}{\sqrt{\det(Q)}} \exp\left(-\frac{1}{2} \begin{pmatrix} N^{-1}(u_1) \\ N^{-1}(u_2) \\ N^{-1}(u_3) \end{pmatrix}^T (Q^{-1} - I) \begin{pmatrix} N^{-1}(u_1) \\ N^{-1}(u_2) \\ N^{-1}(u_3) \end{pmatrix} \right)$$

In general, Q^{-1} will have 3 different off-diagonal elements, derived from the three different correlations between u_1 and u_2, between u_2 and u_3 and between u_3 and $u_1$respectively. Thus the form of the copula density if $u_1 = 0$ expressed as a function of the remaining two components of u, i.e. here u_2 and u_3, will differ from its form if $u_2 = 0$ expressed as a function of u_1 and u_3 etc. However, to be Archimedean, the copula needs to be indifferent between the components of u.

For $n > 2$, the Gaussian copula has too many free parameters to be Archimedean.

Conversely, if returns are multivariate normal and have an exchangeable copula then the returns can be characterised by a factor structure involving a single factor.

A set of m random variables, x_i $(i = 1, \dots, m)$ is said to possess a factor structure if their covariance matrix, V, is of the form $V = AA^T + B$ where V is an $m \times m$ matrix, A is an $m \times k$ matrix (and there are k factors) and B is a diagonal matrix. Suppose the variance of each x_i is σ_i^2 and we define $y_i = x_i / \sigma_i$. Then y_i have unit variance and their covariance (now also correlation) matrix also has the form $\bar{V} = \bar{A}\bar{A}^T + \bar{B}$. The copulas describing the x_i and y_i are the same. If it is exchangeable and x_i are multivariate normal then we must have $corr(x_i, x_j)$ being the same for all $i \neq j$, say $corr(x_i, x_j) = \rho$. This arises if we set A and B as follows, if I is the identity matrix:

$$A = \begin{pmatrix} \sqrt{\rho} \\ \vdots \\ \sqrt{\rho} \end{pmatrix} \qquad and \qquad B = (1 - \rho)I$$