Deriving the principal components of two uncorrelated return series

[Nematrian website page: ERMMTDeriving2DimensionalPrincipalComponents, © Nematrian 2015]

Suppose the returns on two uncorrelated series are $r_{1,t}$ and $r_{2,t}$. It is assumed that we want an analytical solution rather than a numerical solution (a numerical solution can be found using Nematrian web services functions that target principal components, i.e. MnPrincipalComponents, MnPrincipalComponentsSizes and MnPrincipalComponentsWeights).

For a two series problem, if the covariance matrix is $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ then the principal components are associated with the eigenvectors and eigenvalues of the covariance matrix, i.e. with values of λ that satisfy, for some vector $x = (x_1, x_2)^T$ the equation $Vx = \lambda x$. The λ therefore satisfy the following equations:

$$V_{11}x_1 + V_{12}x_2 = \lambda x_1$$
$$V_{21}x_1 + V_{22}x_2 = \lambda x_2$$

This means that $(V_{11} - \lambda)x_1 + V_{12}x_2 = 0$ and $V_{21}x_1 + (V_{22} - \lambda)x_2 = 0$, i.e. (since $V_{12} = V_{21}$ for a covariance matrix):

$$\lambda^2 - (V_{11} + V_{22})\,\lambda + V_{11}V_{22} - {V_{12}}^2 = 0$$

In this instance, the two series are uncorrelated and therefore $V_{12} = V_{21} = 0$. The quadratic then becomes $(\lambda - V_{11})(\lambda - V_{22}) = 0$, i.e. $\lambda = V_{11}$ or $\lambda = V_{22}$. The (population) covariance matrix is $\begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix}$, which thus has two eigenvalues $\lambda_1 = V_{11}$ and $\lambda_2 = V_{22}$ and associated eigenvectors which are of the form $b_1 = (k_1, 0)^T$ and $b_2 = (0, k_2)^T$ respectively for arbitrary k_1 and k_2. The first principal component is associated with whichever of λ_1 and λ_2 is the larger, and the second principal component with the other one.

If we want principal components that are orthonormal return series and portfolio exposures that correspond to these principal components then the portfolio exposures must have $|b_i| = 1$ for $i = 1$ and 2, which means in this instance that $k_1 = \pm 1$ and $k_2 = \pm 1$. If we choose $k_i = 1$ then the resulting return series are merely de-meaned versions of the original series, i.e. are are $\tilde{r}_{1,t} = r_{1,t} - \bar{r}_1$ and $\tilde{r}_{2,t} = r_{2,t} - \bar{r}_2$ or vice-versa depending on whether $\lambda_1 > \lambda_2$ or $\lambda_1 < \lambda_2$.