Derivation of the Cornish-Fisher asymptotic expansion

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A common methodology within risk management circles for estimating the shape of a fat-tailed return distribution is to make use of the Cornish-Fisher asymptotic expansion, see e.g. Abramowitz and Stegun (1970). The Cornish Fisher asymptotic expansion in effect takes into account non-Normality, and thus by implication moments higher than the second moment, by using a formula in which terms in higher order moments explicitly appear. Most commonly the focus is on the fourth-moment version of this expansion, since it merely uses moments up to and including kurtosis. In effect, the fourth-moment Cornish Fisher approach aims to provide a reliable estimate of the distribution's entire quantile-quantile plot merely from the first four moments of the distribution, i.e. its mean, standard deviation, skew and kurtosis.

The approach works as follows. Let Y_i be identically distributed random variables. Let the cumulative distribution function of $Y = \sum_{i=1}^{n} Y_i$ be denoted by $F(y)$. Then the (Cornish-Fisher) asymptotic expansion (with respect to n) for the value of y_p such that $F(y_p) = 1 - p$ is $y_p \sim m + \sigma w$ where:

$$w = x + [\gamma_1 h_1(x)] + [\gamma_2 h_2(x) + \gamma_1^2 h_{11}(x)] + \cdots$$

Here terms in brackets are terms of the same order with respect to n, m is the mean of the distribution, σ the standard deviation of the distribution and κ_r are the distribution's *cumulants*, i.e. the coefficients of the following power series expansion for $\log \phi(t)$ where $\phi(t)$ is the distribution's *characteristic function*.

$$\log \phi(t) = \sum_{n=0}^{\infty} \kappa_r (it)^n / n!$$

The cumulants are related to the moments of the distribution via the relationship $\gamma_{r-2} = \kappa_r / \kappa_2^{r/2}$ (for $r = 3,4,\ldots$) where γ_1 is the skew(ness), γ_2 is the (excess) kurtosis etc. x is the relevant cumulative Normal distribution point, i.e. the value for which:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = p$$

and

$$h_1(x) = \frac{1}{6} He_2(x) \qquad h_2(x) = \frac{1}{24} He_3(x) \qquad h_{11}(x) = -\frac{1}{36}\left(2He_3(x) + He_1(x)\right) \quad \ldots$$

The $He_n(x)$ are the Hermite polynomials, i.e.:

$$He_n(x) = n! \sum_{m=0}^{int(n/2)} \frac{(-1)^m x^{2-2m}}{2^m m! \, (n-2m)!}$$

In effect, if we are using a fourth moment Cornish-Fisher adjustment then this means estimating the shape of a quantile-quantile plot by the following cubic, where γ_1 is the skew and γ_2 is the kurtosis of the distribution:

$$y(x) = m + \sigma\left(x + \frac{\gamma_1(x^2 - 1)}{6} + \frac{3\gamma_2(x^3 - 3x) - 2\gamma_1^2(2x^3 - 5x)}{72}\right)$$

For standardised returns (with $m = 0$ and $\sigma = 1$), this simplifies to:

$$y(x) = x + \frac{\gamma_1(x^2 - 1)}{6} + \frac{3\gamma_2(x^3 - 3x) - 2\gamma_1^2(2x^3 - 5x)}{72}$$