Deriving the Black-Scholes Option Pricing Formulae using the limit of a suitably constructed lattice

Suppose we knew for certain that between time $t - h$ and $t$ the price of the underlying could move from $S$ to either $Su$ or to $Sd$, where $d < u$ (as in the diagram below), that cash (or more precisely the appropriate risk-free asset) invested over that period earns an interest rate (continuously compounded) of $r$ and that the underlying (here assumed to be an equity or an equity index) generates income, i.e. dividend yield, (continuously compounded) of $q$.

Diagram illustrating single time-step binomial option pricing

$$
\begin{align*}
S & \quad \text{or} \quad \text{Su} \\
h & \quad \text{or} \quad Sd
\end{align*}
$$

Suppose that we also have a derivative (or indeed any other sort of security) which (at time $t$) is worth $A = V(Su, t)$ if the share price has moved to $Su$, and worth $B = V(Sd, t)$ if it has moved to $Sd$.

Starting at $S$ at time $t - h$, we can (in the absence of transaction costs and in an arbitrage-free world) construct a hedge portfolio at time $t - h$ that is guaranteed to have the same value as the derivative at time $t$ whichever outcome materialises. We do this by investing (at time $t - h$) $fS$ in $f$ units of the underlying and investing $gS$ in the risk-free security, where $f$ and $g$ satisfy the following two simultaneous equations:

$$
\begin{align*}
fSu e^{qh} + gSe^{rh} &= A = V(Su, t) \\
fSd e^{qh} + gSe^{rh} &= B = V(Sd, t)
\end{align*}
$$

Hence:

$$
\begin{align*}
fS &= e^{-qh} \frac{V(Su, t) - V(Sd, t)}{u - d} \\
gS &= e^{-rh} \frac{-dV(Su, t) + uV(Sd, t)}{u - d}
\end{align*}
$$

The value of the hedge portfolio and hence, by the principle of no arbitrage, the value of the derivative at time $t - h$ can thus be derived by the following backward equation:

$$
V(S, t - h) = fS + gS = \frac{e^{(r-q)h} - d}{u - d} e^{-rh} V(Su, t) + \frac{u - e^{(r-q)h}}{u - d} e^{-rh} V(Sd, t)
$$

We can rewrite this equation as follows, where $p_u = (e^{(r-q)h} - d)/(u - d)$ and $p_d = (u - e^{(r-q)h})/(u - d)$ and hence $p_u + p_d = 1$.

$$
V(S, t - h) = p_u e^{-rh} V(Su, t) + p_d e^{-rh} V(Sd, t)
$$
Assuming that the two potential movements are chosen so that \( p_u \) and \( p_d \) are both positive, i.e. with \( u > e^{(r-q)h} > d \) then \( p_u \) and \( p_d \) correspond to the relevant risk neutral probabilities for the lattice element. Getting \( p_u \) and \( p_d \) to adhere to this constraint is not normally difficult for an option like this since \( e^{(r-q)h} \) is the forward price of the security and it would be an odd sort of binomial tree that did not straddle the expected movement in the underlying.

In the multi-period analogue, the price of the underlying is assumed to be able to move in the first period either up or down by a factor \( u \) or \( d \), and in second and subsequent periods up or down by a further \( u \) or \( d \) from where it had reached at the end of the preceding period. \( u \) or \( d \) can in principle vary depending on the time period (e.g. \( u \) might be size \( u_i \) in time step \( i \), etc.) but it would then be usual to require the lattice to be recombining. In such a lattice an up movement in one time period followed by a down movement in the next leaves the price of the underlying at the same value as a down followed by an up. It would also be common, but again not essential (and sometimes inappropriate), to have each time period of the same length, \( h \).

By repeated application of the backward equation referred to above, we can derive the price \( n \) periods back, i.e. at \( t = T - nh \), of a derivative with an arbitrary payoff at time \( T \). If \( u, d, p_u, p_d, r \) and \( q \) are the same for each period then:

\[
V(S, T - nh) = e^{-rn} \sum_{m=0}^{n} \binom{n}{m} p_u^m p_d^{n-m} V(Su^m d^{n-m}, T)
\]

where:

\[
\binom{n}{m} = \frac{n!}{m!(n-m)!}
\]

This can be re-expressed as an expectation under a risk-neutral probability distribution, i.e. in the following form, where \( E(X|I) \) means the expected value of \( X \) given the risk neutral measure, conditional on being in state \( I \) when the expectation is carried out:

\[
V(S, t) = E(e^{-r(T-t)} V(S, T)|S_t)
\]

Suppose we have a European-style put option with strike price \( K \) (assumed to be at a node of the lattice) maturing at time \( T \) and we want to identify its price, \( P(S, t) \) prior to maturity, i.e. where \( t < T \). Suppose also that \( r \) and \( q \) are the same for each time period. The price of the option at maturity is given by its payoff, i.e. \( P(S, T) = \max(K - S, 0) \) where \( K = S_0 u^{m_0} d^{n-m_0} \) say for some \( m_0 \) (here \( S_0 \) is the price ruling at time \( t = 0 \) used to construct the first node in the lattice). Applying the multi-period pricing formula set out above, we find that the price of such an option at time \( t = T - nh < T \) in such a framework is as follows, where \( B(x, n, p) \) is the binomial probability distribution function, i.e. \( B(x, n, p) = \sum_{m=0}^{x} \binom{n}{m} p^m (1-p)^{n-m} \), bearing in mind that \( p_u + p_d = 1 \):

\[
P(S, T - nh) = e^{-rn} \sum_{m=0}^{m_0} \binom{n}{m} p_u^m p_d^{n-m} (K - S_0 u^m d^{n-m})
\]

\[
\Rightarrow P(S, t) = e^{-r(T-t)} KB(m_0, n, p) - e^{-q(T-t)} B(m_0, n, \frac{up}{up + dp})
\]

Suppose we define the volatility of the lattice to be \( \sigma = \log(u/d)/(2\sqrt{h}) \) and suppose too that this is constant, i.e. the same for each time period. Then if we allow \( h \) to tend to zero, keeping \( \sigma, t, T \)
etc. fixed, with $u/d \to 1$ by, say, setting \( \log(u) = \sigma \sqrt{h} \) and \( \log(d) = -\sigma \sqrt{h} \), we find that the above formula and hence the price of the put option tends to:

\[
P(S, t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)
\]

where

\[
d_1 = \frac{\log(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}
\]

and \( N(z) \) is the cumulative Normal distribution function, i.e.

\[
N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx
\]

The corresponding formula (in the limit) for the price, \( C(S, t) \) of a European call option maturing at time \( T \) with a strike price of \( K \) can be derived in an equivalent manner as:

\[
C(S, t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)
\]

This formula can also be justified on the grounds that the value of a combination of a European put option and a European call option with the same strike price should satisfy so-called put-call parity, if they are to satisfy the principle of no arbitrage, i.e. (after allowing for dividends and interest):

\[
stock + put = cash + call \Rightarrow Se^{-q(T-t)} + P = Ke^{-r(T-t)} + C
\]

Strictly speaking, these formulae for European put and call options are the Garman-Kohlhagen formulae for dividend bearing securities and only if \( q \) is set to zero do they become the original Black-Scholes option pricing formulae, although in practice most people would actually refer to these formulae as the Black-Scholes formulae, and call a world satisfying the assumptions underlying these formulae as a ‘Black-Scholes’ world. The volatility \( \sigma \) used in their derivation has a natural correspondence with the volatility that the share price might be expected to exhibit in a Black-Scholes world.